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Nonlinear field line random walk for non-Gaussian statistics

A Shalchi[1], **J A le Roux, G M Webb and G P Zank**

CSPAR, University of Alabama, Huntsville, AL 35805, USA

E-mail: andreasm4@yahoo.com

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Abstract

We investigate analytically the random walk of magnetic field lines. In previous articles about this subject, a Gaussian model has been used for replacing the field line distribution function. Here we employ a Kappa distribution to investigate the influence of a non-Gaussian statistics. As shown, only the amplitude of the field line mean square deviation and the field line diffusion coefficient are different from the Gaussian model if we assume $\kappa > 2$. It seems that the exact form of the field line distribution is less important for computing field line diffusion coefficients in this case. This conclusion confirms previous investigations performed within the framework of a Gaussian statistics.

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1. Introduction

A fundamental problem of plasma and astrophysics is the random walk of magnetic field lines. In the theory of field line wandering we assume a superposition of a mean magnetic field and a stochastic component. Such configurations can be found in the solar wind or in the interstellar medium. Due to the stochastic component, field lines are not well defined and have to be described using methods of statistical physics. The main aim of the theory of field line random walk (FLRW) is the computation of the field line mean square displacement and the field line distribution function perpendicular to the mean magnetic field.

While the investigation of FLRW can lead to an improved understanding of turbulence, the knowledge of FLRW can also be important for describing the interaction between turbulence and charged particles (e.g., cosmic rays) which experience scattering. As shown by different authors (e.g., Webb *et al* 2006, Shalchi and Kourakis 2007c, Shalchi *et al* 2007, Webb *et al* 2008) field line diffusion coefficients can directly be related to charged particle transport

[1] Permanent address: Institut für Theoretische Physik, Lehrstuhl IV: Weltraum- und Astrophysik, Ruhr-Universität Bochum, D-44780 Bochum, Germany.

parameters describing scattering in the direction perpendicular to the mean magnetic field (e.g., the magnetic field of the Sun).

Already in the 1960s of the 20th century investigators have started to describe the random walk of magnetic field lines (see, e.g., Jokipii and Parker 1969). In these early papers, a one-dimensional model (also known as slab model) was used for approximating the turbulent magnetic fields. In more recent years, however, it became more and more clear that the slab model provides only a very crude approximation for solar wind turbulence (see, e.g., Matthaeus *et al* 1990). Whereas for slab turbulence the field line random walk can be described by an exact formulation, one needs a theory for evaluating the field line statistics for any other model.

Matthaeus *et al* (1995) have developed a nonlinear theory for computing field line diffusion coefficients for the so-called slab/2D composite model. Within the latter model it is assumed that solar wind turbulence can be approximated by a superposition of slab fluctuations and two-dimensional modes. This model seems to be an accurate model for solar wind turbulence (see, e.g., Matthaeus *et al* 1990). Some years later, Shalchi and Kourakis (2007a, 2007b) have generalized the diffusion theory of Matthaeus *et al* (1995) to allow also a non-diffusive behavior of FLRW. For several turbulence spectra these authors found superdiffusive FLRW. Other authors obtained also superdiffusion of field lines (see, e.g., Zimbardo *et al* 1995, 2000).

All these theoretical results are based on the assumption that the field lines can be described by a Gaussian statistics. So far, however, this assumption has not been justified. It is the purpose of the present paper to employ different field line distribution functions to explore the importance of the field line statistics. We compute field line mean square deviations as well as field line diffusion coefficients for a Kappa distribution. From the latter distribution the Gaussian model can be obtained by a limiting process. Furthermore, we derive some general properties of the field line mean square deviation and the field line diffusion coefficient for arbitrary field line statistics.

The Kappa distribution has been successfully used to fit the distribution function of the observed heliospheric magnetic field fluctuations at Earth on scales of 1 h (the timescale associated with the turbulence correlation scale) to nearly 1 year (e.g., Burlaga and Vinas 2004). The data fits show that the magnetic field distribution function deviate strongly from a Gaussian distribution on timescales of about 1 h to about 85 days, because $1.2 < \kappa < 1.5$ (if $\kappa \rightarrow \infty$, the Kappa distribution converges to a Gaussian distribution). This indicates that the inertial and energy range of the power spectrum of magnetic fluctuations in the solar wind is strongly non-Gaussian. The Kappa distribution can be interpreted in terms of the Tsallis distribution representing Tsallis statistics (e.g., Burlaga and Vinas 2004, Leubner and Voros 2005). This statistics is a generalization of the standard Boltzmann (Gaussian) statistics to model the effects of long-range forces, and long-time memory effects caused by intermittency (the presence of coherent nonlinear structures) in systems such as the solar wind plasma. Thus, the use of a Kappa distribution for modeling non-Gaussian statistics in the solar wind has the advantage of a generalized statistical foundation in contrast to other non-Gaussian distributions that have been used to fit observed magnetic field distributions in the solar wind. Although we do not discuss dissipation effects in the turbulence wave spectrum, we like to note that several studies show that non-Gaussian fluctuations are also found at the small scales of the dissipation range (see, e.g., Leamon *et al* 1998, Sorriso-Valvo *et al* 1999, Alexandrova *et al* 2008).

However, the observed κ values as described above, have to do with the distribution function of the magnetic field fluctuations while in the theory of FLRW we compute the distribution of the field line trajectories. Also, the non-Gaussian tails of the observations have to do with intermittency. Sudden jumps in the magnetic field from the edges of nonlinear coherent structures, such as magnetic islands, cause the magnetic field fluctuation distribution function to have strong non-Gaussian tails on short timescales. Coherent magnetic island structures, if dominant, means that the FLRW is subdiffusive and that the distribution function of the FLRW should not have strong extended non-Gaussian power-law tails but probably should cut off. However, it could well be that the random magnetic field lines between the coherent islands are Gaussian or even superdiffusive as the simulations of Zimbardo *et al* (1995, 2000) showed when they solved the nonlinear field line equation numerically. The result of the present paper (superdiffusive FLRW) refers to those fluctuations since the theory of FLRW only deals with random linear wave fluctuations and not with nonlinear coherent structures. Unfortunately, since the observed κ values include the effect of both coherent structures and random fluctuations we do not know at this stage what the κ values are of the random magnetic field component in between the magnetic islands, and we do not know how the kappa value of the magnetic fluctuation distribution relate to the distribution function of the FLRW. In the following sections we, therefore, treat κ as an unknown parameter which can be used for investigating the influence of different field line distributions on the random walk of magnetic field lines.

2. The nonlinear theory for FLRW

2.1. General equations

In this section, we discuss the standard approach for describing field line random walk analytically in magnetostatic turbulence. The equation for the field lines $\vec{x} = (x(z), y(z), z)$ is $\mathrm{d}x B_z(\vec{x}) = \mathrm{d}z B_x(\vec{x})$. If we adopt turbulent magnetic fields with vanishing z-component ($\delta B_z = 0$) and that the mean field is aligned parallel to the z-axis ($\vec{B}_0 = B_0 \vec{e}_z$) the field line equation becomes

$$\mathrm{d}x = \frac{\delta B_x(\vec{x})}{B_0}\,\mathrm{d}z. \tag{1}$$

The solution of this equation provides the field line $x = x(z)$. A similar equation can be found for the y-component. By integrating equation (1) the displacement in the x-direction can be written as

$$\Delta x(z) = \frac{1}{B_0}\int_0^z \mathrm{d}z'\,\delta B_x(\vec{x}(z')) \tag{2}$$

and thus we find for the mean square displacement

$$\langle(\Delta x(z))^2\rangle = \frac{1}{B_0^2}\int_0^z \mathrm{d}z' \int_0^z \mathrm{d}z'' \langle \delta B_x(\vec{x}(z'))\delta B_x^*(\vec{x}(z''))\rangle. \tag{3}$$

By applying a Fourier representation for the magnetic fields, we can easily derive

$$\langle(\Delta x(z))^2\rangle = \frac{1}{B_0^2}\int \mathrm{d}^3k \int \mathrm{d}^3k' \int_0^z \mathrm{d}z' \int_0^z \mathrm{d}z'' \langle \delta B_x(\vec{k})\delta B_x^*(\vec{k}')\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}(z')-\mathrm{i}\vec{k}'\cdot\vec{x}(z'')}\rangle \tag{4}$$

where we have used the ensemble average operator $\langle\cdots\rangle$. To proceed, we have to evaluate the correlation function on the right-hand side of equation (4).

2.2. The nonlinear theory for FLRW

Equation (4) is a nonlinear equation since the field lines $x(z)$ can be found on the right-hand side as well as on the left-hand side. One possibility to evaluate such equations is the application of quasilinear theory (see, e.g., Jokipii and Parker 1969). In this case the field lines on the right-hand side of equation (4) are replaced by $x(z) = y(z) = 0$ corresponding to the unperturbed system ($\delta B_i = 0$). A more appropriate (nonlinear) formulation for field line random walk was proposed by Shalchi and Kourakis (2007a), which is a generalization of the Matthaeus *et al* (1995) theory. Within a nonlinear formulation, we have to apply Corrsin's independence hypothesis (Corrsin 1959, Salu and Montgomery 1977, McComb 1990) in equation (4) to obtain

$$\langle \delta B_x(\vec{k})\delta B_x^*(\vec{k}')\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}(z')-\mathrm{i}\vec{k}'\cdot\vec{x}(z'')}\rangle = \langle \delta B_x(\vec{k})\delta B_x^*(\vec{k}')\rangle\langle \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}(z')-\mathrm{i}\vec{k}'\cdot\vec{x}(z'')}\rangle. \tag{5}$$

As described in Matthaues *et al* (1995), the basic idea of the Corrsin approximation is that the statistics of the magnetic fluctuations can be separated from those of the individual trajectories. This separation reflects the fact that the random trajectory $\vec{x}(z)$ is highly irregular and sensitive to phases of the magnetic fluctuations, and not just the spectrum.

To proceed we assume homogeneous turbulence $\langle \delta B_x(\vec{k})\delta B_x^*(\vec{k}')\rangle = P_{xx}(\vec{k})\delta(\vec{k}-\vec{k}')$ leading to

$$\langle (\Delta x(z))^2\rangle = \frac{1}{B_0^2}\Re\mathfrak{e}\int \mathrm{d}^3k\, P_{xx}(\vec{k})\int_0^z \mathrm{d}z'\int_0^z \mathrm{d}z''\langle \mathrm{e}^{\mathrm{i}\vec{k}\cdot[\vec{x}(z')-\vec{x}(z'')]}\rangle. \tag{6}$$

For homogeneous turbulence the term in the brackets $\langle\cdots\rangle$ depends only on $|z'-z''|$ and, therefore,

$$\Gamma(z',z'') \equiv \Re\mathfrak{e}\langle \mathrm{e}^{\mathrm{i}\vec{k}\cdot[\vec{x}(z')-\vec{x}(z'')]}\rangle = \Gamma(|z'-z''|). \tag{7}$$

In general, one can write

$$\int_0^z \mathrm{d}z'\int_0^z \mathrm{d}z''\,\Gamma(|z'-z''|) = \int_0^z \mathrm{d}z'\int_0^{z'} \mathrm{d}z''\,\Gamma(z'-z'') + \int_0^z \mathrm{d}z'\int_{z'}^z \mathrm{d}z''\,\Gamma(z''-z'). \tag{8}$$

By using the integral transformation $y = z'-z''$ in the first, and $y = z''-z'$ in the second integral, we find

$$\int_0^z \mathrm{d}z'\int_0^z \mathrm{d}z''\,\Gamma(|z'-z''|) = \int_0^z \mathrm{d}z'\int_0^{z'} \mathrm{d}y\,\Gamma(y) + \int_0^z \mathrm{d}z'\int_0^{z-z'} \mathrm{d}y\,\Gamma(y). \tag{9}$$

By inserting $1 = \mathrm{d}z'/\mathrm{d}z'$ in both integrals

$$\int_0^z \mathrm{d}z'\int_0^z \mathrm{d}z''\,\Gamma(|z'-z''|) = \int_0^z \mathrm{d}z'\frac{\mathrm{d}z'}{\mathrm{d}z'}\int_0^{z'} \mathrm{d}y\,\Gamma(y) + \int_0^z \mathrm{d}z'\frac{\mathrm{d}z'}{\mathrm{d}z'}\int_0^{z-z'} \mathrm{d}y\,\Gamma(y) \tag{10}$$

and by using partial integration (p.I.), we have

$$\begin{aligned}\int_0^z \mathrm{d}z'\int_0^z \mathrm{d}z''\,\Gamma(|z'-z''|) &\overset{\text{p.I.}}{=} z\int_0^z \mathrm{d}y\,\Gamma(y) - \int_0^z \mathrm{d}z'\,z'\frac{\mathrm{d}}{\mathrm{d}z'}\int_0^{z'} \mathrm{d}y\,\Gamma(y)\\ &\quad - \int_0^z \mathrm{d}z'\,z'\frac{\mathrm{d}}{\mathrm{d}z'}\int_0^{z-z'} \mathrm{d}y\,\Gamma(y).\end{aligned} \tag{11}$$

From this equation one can easily derive

$$\int_0^z \mathrm{d}z'\int_0^z \mathrm{d}z''\,\Gamma(|z'-z''|) = z\int_0^z \mathrm{d}y\,\Gamma(y) - \int_0^z \mathrm{d}z'\,z'\Gamma(z') + \int_0^z \mathrm{d}z'\,z'\Gamma(z-z'). \tag{12}$$

By using $y = z'$ in the second integral and the transformation $y = z - z'$ in the third integral, we find

$$\begin{aligned}\int_0^z \mathrm{d}z'\int_0^z \mathrm{d}z''\,\Gamma(|z'-z''|) &= z\int_0^z \mathrm{d}y\,\Gamma(y) - \int_0^z \mathrm{d}y\,y\Gamma(y) + \int_0^z \mathrm{d}y(z-y)\Gamma(y)\\ &= 2\int_0^z \mathrm{d}y(z-y)\Gamma(y).\end{aligned} \tag{13}$$

Finally, we find the relation

$$\mathfrak{Re} \int_0^z dz' \int_0^z dz'' \langle e^{i\vec{k}\cdot[\vec{x}(z')-\vec{x}(z'')]}\rangle = 2\mathfrak{Re} \int_0^z dz' (z-z') \langle e^{i\vec{k}\cdot\Delta\vec{x}(z')}\rangle \tag{14}$$

and equation (6) becomes

$$\langle(\Delta x(z))^2\rangle = \frac{2}{B_0^2} \int d^3k\, P_{xx}(\vec{k}) \int_0^z dz' (z-z')\Gamma(z') \tag{15}$$

with $\Gamma(z) = \mathfrak{Re}\langle e^{i\vec{k}\cdot\Delta\vec{x}(z)}\rangle$ and $\Delta\vec{x}(z) = \vec{x}(z) - \vec{x}(0)$.

By differentiating this result with respect to z we find

$$\frac{d}{dz}\langle(\Delta x(z))^2\rangle = \frac{2}{B_0^2} \int d^3k\, P_{xx}(\vec{k}) \int_0^z dz'\, \Gamma(z'). \tag{16}$$

By considering the second derivative of the field line mean square deviation, an ordinary differential equation can be obtained

$$\frac{d^2}{dz^2}\langle(\Delta x(z))^2\rangle = \frac{2}{B_0^2} \int d^3k\, P_{xx}(\vec{k})\Gamma(z). \tag{17}$$

This differential equation is correct for arbitrary turbulence (described by $P_{xx}(\vec{k})$) and arbitrary field line statistics (described by $\Gamma(z)$). To proceed, we have to specify the xx-component of the magnetic correlation tensor $P_{xx}(\vec{k})$ as well as the characteristic function $\Gamma(z)$.

Since the characteristic function $\Gamma(z)$ is not known, we will impose the form of this function. The model for $\Gamma(z)$ depends on the variance $\langle(\Delta x(z))^2\rangle$ and, therefore, on z. The dependence of the variance on z can then be deduced from equation (17). In the following sections, we employ these ideas to compute the field line statistics.

3. The characteristic function

3.1. General properties

For Cartesian coordinates the characteristic function has the form

$$\Gamma(z) = \mathfrak{Re}\langle e^{ik_x x + ik_y y + ik_z z}\rangle. \tag{18}$$

In the theory of field line random walk z is a variable and not a stochastic quantity. Thus, we can write

$$\Gamma(z) = \mathfrak{Re}[e^{ik_z z}\langle e^{ik_x x + ik_y y}\rangle]. \tag{19}$$

For Cartesian coordinates we can use

$$\Gamma(z) = \mathfrak{Re}\left[e^{ik_z z} \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\, f(x,y,z)\, e^{ik_x x + ik_y y}\right] \tag{20}$$

and for cylindrical coordinates

$$\Gamma(z) = \mathfrak{Re}\left[e^{ik_z z} \int_0^{2\pi} d\phi \int_0^{\infty} d\rho\, \rho f(\phi,\rho,z)\, e^{ik_\perp\rho\cos(\Psi)\cos(\phi) + ik_\perp\rho\sin(\Psi)\sin(\phi)}\right]. \tag{21}$$

By assuming axisymmetric turbulence we can use $f(\phi,\rho,z) = f(\rho,z)$ and, therefore,

$$\begin{aligned}\Gamma(z) &= \mathfrak{Re}\left[e^{ik_z z} \int_0^{\infty} d\rho\, \rho f(\rho,z) \int_0^{2\pi} d\phi\, e^{ik_\perp\rho\cos(\phi-\Psi)}\right] \\ &= 2\pi\cos(k_z z) \int_0^{\infty} d\rho\, \rho f(\rho,z) J_0(k_\perp\rho),\end{aligned} \tag{22}$$

where we have used $\cos(\Psi)\cos(\phi) + \sin(\Psi)\sin(\phi) = \cos(\Psi - \phi)$ and (see, e.g., Gradshteyn and Ryzhik 2000)

$$\int_0^{2\pi} \mathrm{d}\phi\, \mathrm{e}^{\mathrm{i}k_\perp \rho \cos(\phi-\Psi)} = 2\pi J_0(k_\perp \rho). \tag{23}$$

In Equations (22) and (23) we have used the Bessel function $J_0(x)$. To evaluate equation (22) we have to specify the field line distribution $f(\rho, z)$. In section 6 of the present paper, we derive some properties of FLRW for arbitrary $f(\rho, z)$.

3.2. *The simplest model: an exponential distribution*

As an introductory example we replace the distribution function $f(\rho, z)$ in equation (22) by an exponential function of the form

$$f(\rho, z) = \frac{3}{2\pi\sigma^2}\, \mathrm{e}^{-\sqrt{3}\rho/\sigma} \tag{24}$$

with the variance $\sigma^2(z) = \langle(\Delta x)^2\rangle$. Note that $\sigma \equiv \sigma(z)$. The factors in equation (24) arise from the normalization condition

$$1 = \int_0^{2\pi} \mathrm{d}\phi \int_0^{\infty} \mathrm{d}\rho\, \rho f(\rho, z) \tag{25}$$

and the definition of the variance

$$\sigma^2 = \int_0^{2\pi} \mathrm{d}\phi \int_0^{\infty} \mathrm{d}\rho\, \rho x^2 f(\rho, z). \tag{26}$$

For this model equation (22) becomes

$$\Gamma(z) = \frac{3\cos(k_z z)}{\sigma^2} \int_0^{\infty} \mathrm{d}\rho\, \rho J_0(k_\perp \rho)\, \mathrm{e}^{-\sqrt{3}\rho/\sigma(z)}. \tag{27}$$

Using (see, e.g., Gradshteyn and Ryzhik 2000)

$$\int_0^{\infty} \mathrm{d}\rho\, \rho J_0(k_\perp \rho)\, \mathrm{e}^{-\sqrt{3}\rho/\sigma} = \frac{\sqrt{3}\sigma^2}{\left(3 + k_\perp^2\sigma^2\right)^{3/2}} \tag{28}$$

we find

$$\Gamma(z) = 3^{3/2}\cos(k_z z)\left(3 + k_\perp^2\sigma^2\right)^{-3/2}. \tag{29}$$

3.3. *The standard approach: a Gaussian statistics*

For a Gaussian statistics we have

$$f(\rho, z) = \frac{1}{2\pi\sigma^2}\, \mathrm{e}^{-\rho^2/(2\sigma^2)} \tag{30}$$

with the variance $\sigma^2 = \langle(\Delta x)^2\rangle$. For this model equation (22) becomes

$$\Gamma(z) = \frac{\cos(k_z z)}{\sigma^2} \int_0^{\infty} \mathrm{d}\rho\, \rho J_0(k_\perp \rho)\, \mathrm{e}^{-\rho^2/(2\sigma^2)}. \tag{31}$$

Using (see, e.g., Gradshteyn and Ryzhik 2000)

$$\int_0^{\infty} \mathrm{d}\rho\, \rho J_0(k_\perp \rho)\, \mathrm{e}^{-\rho^2/(2\sigma^2)} = \sigma^2\, \mathrm{e}^{-\frac{1}{2}k_\perp^2\sigma^2} \tag{32}$$

we find

$$\Gamma(z) = \cos(k_z z)\, \mathrm{e}^{-\frac{1}{2}k_\perp^2\sigma^2}. \tag{33}$$

This result was combined with equation (17) in several previous articles about FLRW (see, e.g., Matthaeus *et al* 1995, Shalchi and Kourakis 2007a).

3.4. New results: a Kappa distribution of field lines

An alternative model is the Kappa distribution also known as generalized Lorentzian function. This model is very convenient to model observed velocity distributions (see Vasyliunas 1968), since it is quasi-Maxwellian at low and thermal energies, while its non-thermal tail decreases as a power law at high energies, as generally observed in space plasmas; this is in line with the fact that particles of higher energy have larger mean free paths and are thus less likely to achieve partial equilibrium.

In the present paper, we employ such a model for approximating the field line distribution function:

$$f(\rho, z) = a(1 + b^2\rho^2)^{-\kappa}. \tag{34}$$

The parameter κ is a free parameter that can be used to obtain different distributions. The parameter a can be obtained from the normalization condition

$$\begin{aligned} 1 &= \int_0^{2\pi} \mathrm{d}\phi \int_0^{\infty} \mathrm{d}\rho\, \rho f(\rho, z) \\ &= 2\pi a \int_0^{\infty} \mathrm{d}\rho\, \rho (1 + b^2\rho^2)^{-\kappa} \\ &= \frac{\pi a}{b^2(\kappa - 1)} \end{aligned} \tag{35}$$

leading to

$$a = \frac{\kappa - 1}{\pi} b^2. \tag{36}$$

The parameter b can be expressed by the mean square deviation

$$\begin{aligned} \sigma^2 &= \int_0^{2\pi} \mathrm{d}\phi \int_0^{\infty} \mathrm{d}\rho\, \rho x^2 f(\rho, z) \\ &= a \int_0^{2\pi} \mathrm{d}\phi \int_0^{\infty} \mathrm{d}\rho\, \rho^3 \cos^2(\phi)(1 + b^2\rho^2)^{-\kappa} \\ &= \pi a \int_0^{\infty} \mathrm{d}\rho\, \rho^3 (1 + b^2\rho^2)^{-\kappa} \\ &= \frac{\pi a}{2b^4(\kappa - 2)(\kappa - 1)} \\ &= \frac{1}{2b^2(\kappa - 2)} \end{aligned} \tag{37}$$

leading to

$$b^2 = \frac{1}{2\sigma^2(\kappa - 2)}. \tag{38}$$

To obtain a normalized form of the distribution function and to obtain a positive finite variance we have to employ the restriction $\kappa > 2$. The Gaussian model used in previous articles about FLRW can be recovered by the limiting process $\kappa \to \infty$. In figure 1, we have compared the different field line distribution functions.

For the Kappa distribution equation (22) becomes

$$\Gamma(z) = 2\pi a \cos(k_z z) \int_0^{\infty} \mathrm{d}\rho\, \rho J_0(k_\perp \rho)(1 + b^2\rho^2)^{-\kappa}. \tag{39}$$

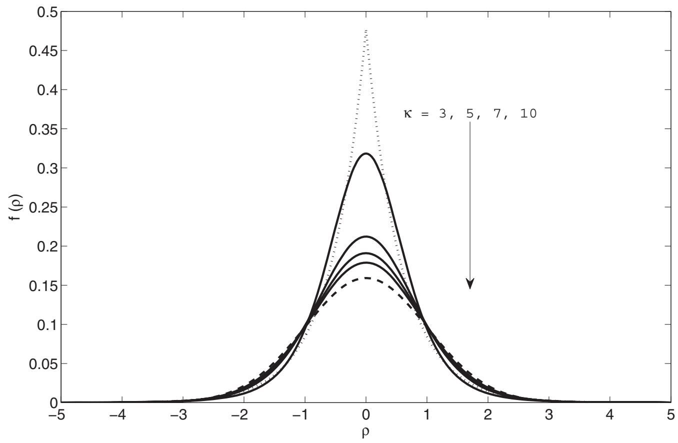

Figure 1. Different distribution functions for $\sigma = 1$. Shown are a simple exponential model (dotted line), the Gaussian model (dashed line) and the Kappa distribution for different values of κ, namely $\kappa = 3, 5, 7, 10$. For $\kappa \to \infty$ the Kappa distribution becomes a Gaussian function.

The integral can be solved (see, e.g., Gradshteyn and Ryzhik 2000)

$$\int_0^\infty \mathrm{d}\rho\, \rho J_0(k_\perp \rho)(1 + b^2\rho^2)^{-\kappa} = \frac{2^{1-\kappa} b^{-1-\kappa} k_\perp^{\kappa-1}}{\Gamma(\kappa)} K_{\kappa-1}\left(\frac{k_\perp}{b}\right), \tag{40}$$

where we used the modified Bessel function of imaginary argument, $K_\lambda(z)$. Therefore, the characteristic function for the Kappa distribution becomes

$$\Gamma(z) = \alpha(\kappa)\cos(k_z z)\left(\frac{k_\perp}{b}\right)^{\kappa-1} K_{\kappa-1}\left(\frac{k_\perp}{b}\right) \tag{41}$$

with

$$\alpha(\kappa) = \frac{2^{2-\kappa}}{\Gamma(\kappa-1)}. \tag{42}$$

In the last step, we have used $\Gamma(z+1) = z\Gamma(z)$ (see, e.g., Abramowitz and Stegun 1974).

4. The field line mean square deviation for a Kappa distribution

The Gaussian model was used in several previous papers about FLRW. Here we focus on the Kappa distribution. By combining Equations (17) and (41) we can describe analytically the random walk of magnetic field lines.

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\sigma^2 = \alpha(\kappa)\frac{2}{B_0^2}\int \mathrm{d}^3k\, P_{xx}(\vec{k})\cos(k_z z)\left(\frac{k_\perp}{b}\right)^{\kappa-1} K_{\kappa-1}\left(\frac{k_\perp}{b}\right). \tag{43}$$

To proceed we have to specify the xx-component of the magnetic correlation tensor.

4.1. Two-dimensional turbulence

It was often suggested (see, e.g., Matthaeus *et al* 1990) that solar wind turbulence can be approximated by a superposition of slab and two-dimensional fluctuations. FLRW for slab turbulence is not very interesting since the theory is exact in this case and does not depend on

Table 1. Turbulence parameters used in the present paper.

Parameter	Physical meaning
s	Inertial range spectral index
q	Energy range spectral index
$C(s, q)$	Normalization function for general $q > -1$
l_{2D}	2D bendover scale
δB_{2D}^2	Magnetic energy of the 2D fluctuations
B_0	Mean magnetic field

any assumptions about the field line distribution. Therefore, we employ a two-dimensional model in the current paper. In this case we have

$$P_{lm}^{2D}(\vec{k}) = g^{2D}(k_\perp)\frac{\delta(k_\parallel)}{k_\perp}\left[\delta_{lm} - \frac{k_l k_m}{k^2}\right], \qquad l, m = x, y \tag{44}$$

leading to

$$\frac{d^2}{dz^2}\sigma^2 = \alpha(\kappa)\frac{2\pi}{B_0^2}\int_0^\infty dk_\perp\, g^{2D}(k_\perp)\left(\frac{k_\perp}{b}\right)^{\kappa-1} K_{\kappa-1}\left(\frac{k_\perp}{b}\right). \tag{45}$$

This is the general formula for field line wandering in two-dimensional turbulence and a Kappa distribution. In the following paragraphs, we specify the wave spectrum $g^{2D}(k_\perp)$.

4.2. An arbitrary turbulence spectrum for the two-dimensional modes

In the current paragraph we employ a spectrum of the form

$$g^{2D}(k_\perp) = \frac{2C(s, q)}{\pi}\delta B_{2D}^2 l_{2D}\frac{(k_\perp l_{2D})^q}{[1 + (k_\perp l_{2D})^2]^{(s+q)/2}} \tag{46}$$

with the normalization function

$$C(s, q) = \frac{\Gamma\left(\frac{s+q}{2}\right)}{2\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}. \tag{47}$$

This spectrum was originally introduced by Shalchi and Weinhorst (2009). The parameters used here are listed in table 1.

The spectrum is correctly normalized for $s > 1$ and $q > -1$. The spectrum is decreasing in the inertial range $\left(k_\perp \geqslant l_{2D}^{-1}\right)$ where the spectrum has the form $\sim k_\perp^{-s}$. For instance, if $s = 5/3$ we can reproduce a Kolmogorov (1941) spectrum and for $s = 3/2$ a Kraichnan (1965) spectrum. We can reproduce an increasing (positive q) and decreasing (negative q) spectrum in the energy range $\left(k_\perp < l_{2D}^{-1}\right)$. The only limitation for the energy range spectral index is $q > -1$. For this spectrum equation (45) becomes

$$\frac{d^2}{dz^2}\sigma^2 = 4C(s, q)l_{2D}\alpha(\kappa)\frac{\delta B_{2D}^2}{B_0^2}\int_0^\infty dk_\perp\frac{(k_\perp l_{2D})^q}{[1 + (k_\perp l_{2D})^2]^{(s+q)/2}}\left(\frac{k_\perp}{b}\right)^{\kappa-1} K_{\kappa-1}\left(\frac{k_\perp}{b}\right). \tag{48}$$

To proceed we employ the integral transformation $x = k_\perp/b$. Furthermore, we use the parameter

$$\beta = b l_{2D} = \frac{1}{\sqrt{2(\kappa - 2)}}\frac{l_{2D}}{\sigma} \tag{49}$$

to find

$$\frac{d^2}{dz^2}\sigma^2 = 4C(s, q)\alpha(\kappa)\frac{\delta B_{2D}^2}{B_0^2}\beta^{q+1} J(\beta, \kappa, s, q) \tag{50}$$

with

$$J(\beta, \kappa, s, q) = \int_0^\infty \mathrm{d}x \frac{x^{q+\kappa-1}}{[1+(\beta x)^2]^{(q+s)/2}} K_{\kappa-1}(x). \tag{51}$$

By considering the limit $z \to \infty$ (stable regime) we expect $\sigma^2 \to \infty$ and, therefore, $\beta \to 0$. In this limit we can use the approximation

$$J(\beta \to 0, \kappa, q) \approx \int_0^\infty \mathrm{d}x\, x^{q+\kappa-1} K_{\kappa-1}(x). \tag{52}$$

The remaining integral can be solved as (see, e.g., Gradshteyn and Ryzhik 2000)

$$J(\beta \to \infty, \kappa, q) \approx 2^{\kappa+q-2}\Gamma\left(\frac{q+2\kappa-1}{2}\right)\Gamma\left(\frac{q+1}{2}\right). \tag{53}$$

With this result equation (50) becomes

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}z^2}\sigma^2 &= 4C(s,q)\alpha(\kappa)\frac{\delta B_{2D}^2}{B_0^2}2^{\kappa+q-2}\Gamma\left(\frac{q+2\kappa-1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)\beta^{q+1} \\ &= F\left(s,q,\kappa,\frac{\delta B_{2D}^2}{B_0^2}\right)\left(\frac{l_{2D}}{\sigma}\right)^{q+1} \end{aligned} \tag{54}$$

with

$$\begin{aligned} F &= F\left(s,q,\kappa,\frac{\delta B_{2D}^2}{B_0^2}\right) \\ &= 2^{(q+3)/2}C(s,q)\frac{\delta B_{2D}^2}{B_0^2}\frac{\Gamma\left(\frac{q+1}{2}\right)\Gamma\left(\frac{q+2\kappa-1}{2}\right)}{\Gamma(\kappa-1)}\left(\frac{1}{\kappa-2}\right)^{(q+1)/2}. \end{aligned} \tag{55}$$

4.3. *Superdiffusion for $q < 1$*

For $q < 1$ this ordinary differential equation can easily be solved by the *Ansatz*

$$\sigma^2 = c|z|^d \tag{56}$$

leading to

$$\begin{aligned} c &= \left[F \cdot l_{2D}^{q+1}\frac{(q+3)^2}{4(1-q)}\right]^{2/(q+3)} \\ d &= 4/(q+3). \end{aligned} \tag{57}$$

Obviously we find (so long as $q < 1$) a superdiffusive behavior of FLRW. The shape of the field line distribution (described by the parameter κ) does not have an influence on the long distance behavior of the variance (described by the parameter d).

In the case $q > 1$ we find a diffusive behavior of FLRW (see Shalchi and Weinhorst 2009). This case is investigated in section 5. In the following paragraph we consider different limits to simplify equation (57).

4.4. *Special cases*

Here we consider different special cases for the parameters κ and q.

4.4.1. Recovery of the Gaussian result. From equation (57) we can recover the Gaussian statistics by employing the limiting process $\kappa \to \infty$. To explore this limit we can use the relation (see, e.g., Abramowitz and Stegun 1974)

$$\lim_{z\to\infty} \frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b}. \tag{58}$$

For $\kappa \to \infty$ and by employing equation (58), equation (57) becomes

$$c = \left[2^{(q+3)/2} C(s,q) \frac{\delta B_{2D}^2}{B_0^2} \Gamma\left(\frac{q+1}{2}\right) l_{2D}^{q+1} \frac{(q+3)^2}{4(1-q)}\right]^{2/(q+3)}$$
$$d = 4/(q+3). \tag{59}$$

Equation (59) is the result for FLRW for the arbitrary spectrum but for Gaussian statistics.

4.4.2. The flat spectrum $q = 0$. Here we simplify equation (57) for the special case $q = 0$ corresponding to a flat spectrum in the energy range

$$c = \left[9C(q=0,s)\sqrt{\frac{\pi}{2}} l_{2D} \frac{\delta B_{2D}^2}{B_0^2} \frac{\Gamma(\kappa - 1/2)}{\Gamma(\kappa-1)\sqrt{\kappa-2}}\right]^{2/3}$$
$$d = 4/3. \tag{60}$$

Again the Gaussian result can be obtained by the limiting process $\kappa \to \infty$ and by employing equation (58) to find

$$c = \left[9C(q=0,s)\sqrt{\frac{\pi}{2}} l_{2D} \frac{\delta B_{2D}^2}{B_0^2}\right]^{2/3}$$
$$d = 4/3. \tag{61}$$

This result is in agreement with the result derived by Shalchi and Kourakis (2007a). Equations (60) and (61) can now be compared to investigate the influence of the parameter κ. Obviously the exponent d does not depend on the field line statistics described by the parameter κ, only the amplitude c is different. We find from equations (60) and (61)

$$\frac{c_{\text{Kappa}}}{c_{\text{Gauss}}} = \left[\frac{\Gamma(\kappa - 1/2)}{\Gamma(\kappa-1)\sqrt{\kappa-2}}\right]^{2/3}. \tag{62}$$

This ratio is visualized in figure 2. For $\kappa > 3$ the result derived by employing a Kappa distribution agrees with the Gaussian result. Only for $\kappa \approx 2$ the amplitude is much larger.

5. Diffusion theory for the Kappa distribution

Here we investigate the case $q > 1$ by employing a diffusion theory. We start with equation (16) and use the diffusion *Ansatz*

$$\langle(\Delta x(z))^2\rangle = 2|z|D \tag{63}$$

with the field line diffusion coefficient D. Equation (16) becomes for diffusive FLRW

$$D = \frac{1}{B_0^2}\int d^3k\, P_{xx}(\vec{k}) \int_0^\infty dz\, \Gamma(z). \tag{64}$$

Here we have also assumed that the z-integral is convergent. Equation (63) can also be combined with equation (41) to find

$$\Gamma(z) = \alpha(\kappa)\cos(k_z z)\left(\frac{k_\perp}{h}\right)^{\kappa-1} K_{\kappa-1}\left(\frac{k_\perp}{h}\right) \tag{65}$$

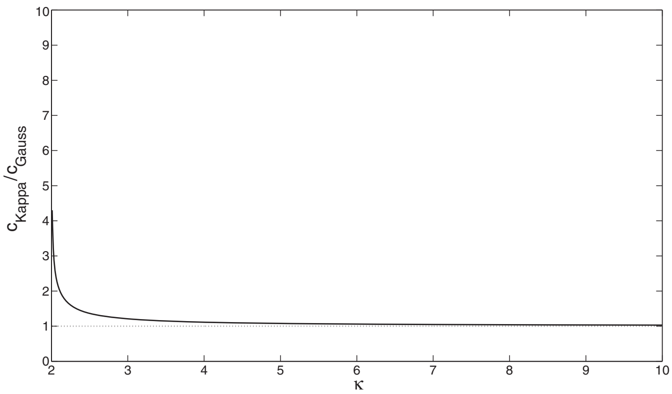

Figure 2. The ratio $c_{\mathrm{Kappa}}/c_{\mathrm{Gauss}}$ versus the parameter κ (solid line). The latter parameter describes the field line statistics. For a Gaussian model (dotted line) this ratio is 1. For $\kappa \gg 2$ the amplitude of the diffusion coefficient corresponds to the Gaussian result. Only for $\kappa \approx 2$ the amplitude is much larger.

with

$$h^2 = \frac{1}{4D|z|(\kappa - 2)}. \tag{66}$$

To proceed we employ a slab/2D composite model for which equation (64) becomes

$$D = D_{\mathrm{slab}} + \frac{1}{B_0^2} \int \mathrm{d}^3k \, P_{xx}^{\mathrm{2D}}(\vec{k}) \int_0^\infty \mathrm{d}z \, \Gamma^{\mathrm{2D}}(z). \tag{67}$$

D_{slab} is the diffusion coefficient of the slab modes. This parameter is not discussed in the present paper since the computation of D_{slab} is straightforward and independent of any theory or assumption of the field line statistics. The z-integral in equation (67) can be rewritten as

$$\begin{aligned}\int_0^\infty \mathrm{d}z \, \Gamma^{\mathrm{2D}}(z) &= \alpha(\kappa) \int_0^\infty \mathrm{d}z \left(\frac{k_\perp}{h}\right)^{\kappa-1} K_{\kappa-1}\left(\frac{k_\perp}{h}\right) \\ &= \frac{2\alpha(\kappa)}{\gamma} \int_0^\infty \mathrm{d}y \, y^\kappa K_{\kappa-1}(y). \end{aligned} \tag{68}$$

Here we have employed the integral transformation $y = \sqrt{\gamma z}$ and we have used the parameter $\gamma = 4D(\kappa - 2)k_\perp^2$. The integral in equation (68) can be solved as (see, e.g., Gradshteyn and Ryzhik 2000)

$$\int_0^\infty \mathrm{d}y \, y^\kappa K_{\kappa-1}(y) = 2^{\kappa-1}\Gamma(\kappa) \tag{69}$$

to find

$$\begin{aligned}\int_0^\infty \mathrm{d}z \, \Gamma^{\mathrm{2D}}(z) &= \frac{2^\kappa \Gamma(\kappa)\alpha(\kappa)}{\gamma} \\ &= \frac{2^\kappa \Gamma(\kappa)\alpha(\kappa)}{4D(\kappa - 2)k_\perp^2} \\ &= \frac{\kappa - 1}{\kappa - 2} \frac{1}{Dk_\perp^2}. \end{aligned} \tag{70}$$

In the last step we have used equation (42). Therefore, we can rewrite equation (67) as

$$D = D_{\mathrm{slab}} + \frac{D_{\mathrm{2D}}^2}{D} \tag{71}$$

with the parameter (diffusion coefficient of the two-dimensional modes)

$$\begin{aligned} D_{\mathrm{2D}}^2 &= \frac{\kappa - 1}{\kappa - 2} \frac{1}{B_0^2} \int \mathrm{d}^3k\, P_{xx}^{\mathrm{2D}}(\vec{k}) k_\perp^{-2} \\ &= \frac{\kappa - 1}{\kappa - 2} \frac{\pi}{B_0^2} \int_0^\infty \mathrm{d}k_\perp\, g^{\mathrm{2D}}(\vec{k}) k_\perp^{-2}. \end{aligned} \tag{72}$$

Equation (71) can be written as

$$D = \frac{D_{\mathrm{slab}} + \sqrt{D_{\mathrm{slab}}^2 + 4D_{\mathrm{2D}}^2}}{2} \tag{73}$$

in agreement with the formula derived by Matthaeus *et al* (1995) for a Gaussian distribution of the field lines. Whereas equation (73) is the same for the Kappa distribution, equation (72) is now slightly different. However, the relation to the wave spectrum is the same; only numerical factors are different.

From equation (72) we can derive the Gaussian result by investigating the limit $\kappa \to \infty$ leading to

$$D_{\mathrm{2D,Gauss}}^2 = \frac{1}{B_0^2} \int \mathrm{d}^3k\, P_{xx}^{\mathrm{2D}}(\vec{k}) k_\perp^{-2} \tag{74}$$

which is the result which was derived in previous articles (see, e.g., Matthaeus *et al* 1995, Shalchi and Weinhorst 2009). The difference between the result obtained for a Kappa distribution and Gaussian statistics is given by the ratio

$$\frac{D_{\mathrm{2D,Kappa}}}{D_{\mathrm{2D,Gauss}}} = \sqrt{\frac{\kappa - 1}{\kappa - 2}}. \tag{75}$$

Although trivial, this ratio is visualized in figure 3. As shown by Shalchi and Weinhorst (2009) FLRW is indeed diffusive for the case $q > 1$. Analytical results for $D_{\mathrm{2D,Gauss}}$ can also be found in Shalchi and Weinhorst (2009). For $q < 1$ we are in the superdiffusive regime which is discussed in section 4.

6. General results for an arbitrary distribution function

Here we investigate equation (17) for arbitrary field line statistics in the case of pure two-dimensional turbulence. In the first paragraph, we compute the field line variance for a constant spectrum at large scales corresponding to $q = 0$. In the second paragraph, we employ the diffusion theory for an arbitrary spectrum.

6.1. Two-dimensional turbulence and a constant spectrum

For two-dimensional turbulence equation (17) becomes

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}z^2}\sigma^2 &= \frac{2\pi}{B_0^2} \int_0^\infty \mathrm{d}k_\perp\, g^{\mathrm{2D}}(k_\perp) \Gamma(z) \\ &= \frac{2\pi}{B_0^2} \int_0^\infty \mathrm{d}k_\perp\, g^{\mathrm{2D}}(k_\perp) \Gamma(\sigma k_\perp). \end{aligned} \tag{76}$$

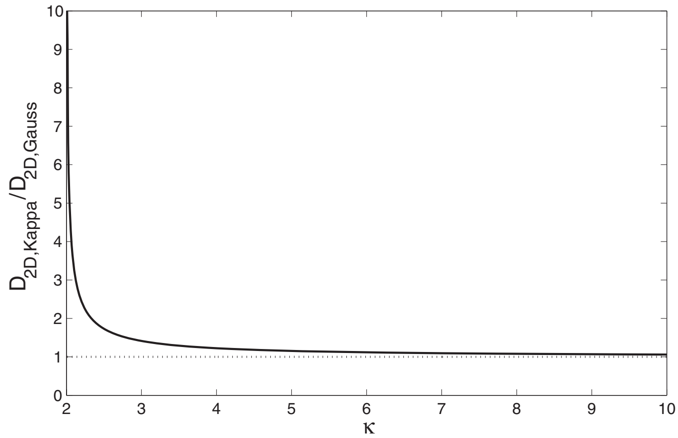

Figure 3. The ratio $D_{2D,Kappa}/D_{2D,Gauss}$ versus the parameter κ (solid line). The latter parameter describes the field line statistics. For a Gaussian statistics (dotted line) this ratio is 1. For $\kappa \gg 2$ the amplitude of the diffusion coefficient corresponds to the Gaussian result. Only for $\kappa \approx 2$ the amplitude is much larger.

In the last step we have assumed that the characteristic function depends only on the dimensionless parameter $\sigma k_\perp$. We are interested in the stable regime of FLRW and, thus, it is reasonable to investigate the limit $\sigma \to \infty$. To evaluate equation (76) we employ the integral transformation $y = k_\perp \sigma$ to find

$$\frac{d^2}{dz^2}\sigma^2 = \frac{2\pi}{B_0^2}\frac{1}{\sigma}\int_0^\infty dy\, g^{2D}\left(y\frac{l_{2D}}{\sigma}\right)\Gamma(y). \tag{77}$$

For the limit $\sigma \to \infty$ and using a constant spectrum in the energy range ($q = 0$) we find

$$\frac{d^2}{dz^2}\sigma^2 \approx \frac{2\pi}{B_0^2}\frac{g^{2D}(0)}{\sigma}\int_0^\infty dy\,\Gamma(y). \tag{78}$$

The y-integral can be expressed using equation (22)

$$\begin{aligned}\int_0^\infty dy\,\Gamma(y) &= 2\pi\int_0^\infty d\rho\,\rho f(\rho,z)\int_0^\infty dy\,J_0\left(y\frac{\rho}{\sigma}\right)\\ &= 2\pi\sigma\int_0^\infty d\rho\, f(\rho,z)\int_0^\infty dx\, J_0(x)\\ &= 2\pi\sigma\int_0^\infty d\rho\, f(\rho,z).\end{aligned} \tag{79}$$

In the last step we have used that the x-integral yields one (see, e.g., Gradshteyn and Ryzhik 2000). To proceed we consider some general properties of the distribution function $f(\rho,z)$. First we assume that $f(\rho,z)$ has the form

$$f(\rho,z) = \frac{f_0}{\sigma^2}h\left(\frac{\rho}{\sigma}\right). \tag{80}$$

This form results from the normalization condition

$$\begin{aligned}1 &= 2\pi\int_0^\infty d\rho\,\rho f(\rho,z)\\ &= 2\pi f_0\int_0^\infty \frac{d\rho}{\sigma}\frac{\rho}{\sigma}h\left(\frac{\rho}{\sigma}\right)\\ &= 2\pi f_0\int_0^\infty dx\, x h(x).\end{aligned} \tag{81}$$

Therefore, f_0 is a parameter which does not depend on the variance σ. By combining equations (79) and (80) we find

$$\int_0^\infty \mathrm{d}y\, \Gamma(y) = 2\pi f_0 \int_0^\infty \frac{\mathrm{d}\rho}{\sigma} h\left(\frac{\rho}{\sigma}\right)$$
$$= 2\pi f_0 \int_0^\infty \mathrm{d}z\, h(z) =: \gamma_D. \tag{82}$$

The parameter γ_D introduced here is a numerical factor which depends on the shape of the distribution function but not on the variance σ. With this result equation (78) becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\sigma^2 = \frac{2\pi}{B_0^2}\gamma_D \frac{g^{2\mathrm{D}}(0)}{\sigma}. \tag{83}$$

Again we can solve this ordinary differential equation by employing the *Ansatz*

$$\sigma^2 = c|z|^d \tag{84}$$

leading to

$$c = \left[\frac{9\pi}{2B_0^2}\gamma_D g^{2\mathrm{D}}(0)\right]^{2/3}$$
$$d = 4/3. \tag{85}$$

The z-dependence of the variance does not depend on the form of the distribution function. Consequently the form of the distribution function does only influence the amplitude of the variance in the form of the parameter γ_D.

6.2. *Diffusion theory for an arbitrary spectrum*

From equation (64) we can derive the field line diffusion coefficient for two-dimensional turbulence and arbitrary field line statistics

$$D = \frac{\pi}{B_0^2}\int_0^\infty \mathrm{d}k_\perp\, g^{2\mathrm{D}}(k_\perp) \int_0^\infty \mathrm{d}z\, \Gamma(z). \tag{86}$$

To proceed we assume again the form $\Gamma(z) = \Gamma(\sigma k_\perp)$ for the characteristic function. By employing the diffusion *Ansatz* $\sigma^2 = 2zD$ this becomes $\Gamma(z) = \Gamma(\sqrt{2zD}k_\perp)$. Using the integral transformation $y = \sqrt{2zD}k_\perp$ we find

$$\int_0^\infty \mathrm{d}z\, \Gamma(z) = \int_0^\infty \mathrm{d}z\, \Gamma(\sqrt{2zD}k_\perp)$$
$$= \frac{1}{Dk_\perp^2}\int_0^\infty \mathrm{d}y\, y\Gamma(y)$$
$$= \frac{\epsilon_D}{Dk_\perp^2}. \tag{87}$$

The parameter ϵ_D does not depend on the wave number $k_\perp$ or the field line diffusion coefficient D, but on the shape of the distribution function. With this form equation (86) becomes

$$D^2 = \frac{\pi\epsilon_D}{B_0^2}\int_0^\infty \mathrm{d}k_\perp\, g^{2\mathrm{D}}(k_\perp)k_\perp^{-2} \tag{88}$$

or, in terms of the diffusion coefficient obtained by assuming a Gaussian statistics

$$D = \sqrt{\epsilon_D} D_{\mathrm{Gauss}}. \tag{89}$$

For all possible distributions we obtain the same form of the field line diffusion coefficient D, only the amplitudes depend on the assumed statistics.

7. Summary and conclusion

In the present paper, we have revisited the problem of the field line random walk. For any turbulence model except the case of pure slab fluctuations one has to employ a nonlinear theory for describing the field line statistics. Furthermore, the field line distribution function has to be specified in order to evaluate the nonlinear theory of field line wandering. In previous articles a Gaussian distribution function has been employed also in cases for which a nondiffusive behavior of FLRW was obtained. This can be seen as one of the weaknesses of the nonlinear description of field line wandering.

Therefore, we have investigated the influence of non-Gaussian statistics on the FLRW in the present paper. We derived analytically the field line mean square displacement (for the non-diffusive case) and the field line diffusion coefficient D for a Kappa distribution. Furthermore, we have derived results for two-dimensional turbulence but arbitrary statistics. In all cases we have demonstrated that the assumed statistics has only an influence on the amplitude of the field line mean square displacement or the field line diffusion coefficient.

This result justifies previous results obtained for Gaussian statistics. It seems that the nonlinear standard theory for FLRW (see, e.g., Matthaeus *et al* 1995, Shalchi and Kourakis 2007a) is indeed qualitatively correct even if the real field line distribution function is not in agreement with a Gaussian function. In the case of a Kappa distribution function we only obtain different amplitudes for the field line mean square displacement and the field line diffusion coefficient if $\kappa \approx 2$. In this case the amplitudes are much larger in comparison to the Gaussian result.

The main result of this work is that the form of the field line distribution function does not influence the qualitative character of transport. This result is at variance with some previous articles on anomalous diffusion, where it is shown that non-Gaussian, power-law distributions in space are closely linked to superdiffusion and Levy random walks (see for instance the recent review by Metzler and Klafter (2004)). A possible explanation for this variance could be the assumption $\kappa > 2$ used in the present paper to ensure a finite variance.

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